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Mathematical Derivative for Solving Constrained Optimization in Utility Functions in Economics

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ABSTRACT

Mathematics is an important assistant tool used by other disciplines to obtain quantitative result in the problem-solving process. The optimum problem is one of the exciting problems that will be the object of our discussion in this article. The main idea of this article is to show how Substitution and Lagrange multiplier methods can solve a constrained optimization problem in economics. In this article, author will study and show how far and easy to solve economic optimum problems using mathematical formulas. Then, with an example of daily problem in economics, the author hopes that it will provide an overview for economic practitioners about how potential mathematical concepts can help them. And it will also be understood that how important it is, to approach and simplify a real-world problem in economics with a mathematical model so that mathematical formulas can be applied to appropriately solve economic problems with the best or optimal results. And eventually, it will also show us that economics in fact are increasingly mathematical.

***Keywords**: Substitution Method; Lagrange Multiplier Method; KKT.*

INTRODUCTION

Efficient problem-solving efforts will always direct humans to solve their daily problems appropriately. This effort particularly is to find the best solution in economic and business issues. We need to affort efficiently and consider all alternative solutions with the limited available resources (in the form of funds, time, workforce, and natural resources). This finite resource will always become a constraint in the calculation and decision-making process for acheiving an optimal or best solution.

On the other hand, with the rapid development of mathematics world, many economic and business problems arise can be solved by mathematical theories or formulas, including the issue of achieving efficiency with various contraints as mentioned above. In mathematical economics, the optimization to achieve this efficiency is known as the optimization problem. Furthermore, a constraint in this problem introduces the constraining factors in the calculation of utility functions (Chiang, 1982), that will constrain every human’s effort in obtaining maximum result in real-world situation, especially in economics or business world.

In mathematics, solving an optimization problem can result in integer numbers or fractional/rational numbers, but in real-world situation, the problem frequently needs a solution with integer numbers. In real-world situation, for optaining an optimal solution, this numerical or number calculation sometimes can also become the further limitation of mathematics calculation applied in general in economics or business world. On the other hand, fortunately, mathematical economics will be able to solve this real limitation with integer programming that cannot be completely presented in this article.

Solving-problem in optimization problems in economics or business cannot be done without using mathematical formulas, especially formulas in derivative/differential. The problem will be solved by using one or more derivative formulas if-and-only-if after the real problems translated or formulated into mathematical programs/equations like equations in linear or non-linear programming, even in dynamic mathematical programming.

A utility function or equation in economics is actually an example of mathematical equation forms. Economists can make a utility equation from data in the market or data acquired in real-world situations. And in the constrained optimization problem, economists must also make the constraint functions. Although the utility function could prevail nationally in a certain country, but its constraint function maght differ from city to city. And it is necessary to know that constraint functions can inhibit businessmen for obtaining maximal results (as they wish). Therefore, business men cannot obtain optimal result if they just apply non-constraint optimization calculations in their actual market. This real market situation perforces economist and businessman include constraint function in their utility function calculation if they want to obtain optimal results in market.

Specifically, the optimization problems are full of constraints, which we call the scope of constrained optimization. This constrained optimization can be solved by using derivative formulas in mathematics. To what extent can derivative formula solve the optimization problem in utility function with constraint?

A simple example:

> *A constrained problem that can be solved using derivative formulas is a group of customers who face the utility function $U = 3Q_1Q_2 + 3Q_2$ with Q_1 (first product) and Q_2 (second product). $20Q_1 + 10Q_2 = 100$ is a constraint function such as the customers' budget that must be considered in acheiving maximum utility. Due to the limited budget (i.e., 100) they have, the customers group cannot buy the first product (price: 20) and the second product (price: 10) indefinitely (in total) to acheive utility (maximum satisfaction). (Yaqin, 2008)*

As described in the example above, we can understand that constrained optimization is a real problem that we will always encounter in daily life, both in the production process and in the process of enjoying the products.

Before using derivative formulas to solve it and seeing that the problem is still simple (there are only two variables, Q_1 and Q_2), it is also necessary to reformulate and scope the issue along with the constraints using the following methods:

- Substitution method
- Lagrange multiplier method
- KKT requirements

RESEARCH METHOD

This research is library research for the application of mathematical concept and formulas in economics and business world. Conducting library research in mathematics in this research study involved quantitative calculation and qualitative of a systematic approach to gather, evaluate, and synthesize information of problem-solving results. Below is a step-by-step guide tailored for this mathematical research:

Define Research Question

The topic of interest in this research transformed in the research question as stated above. The research question as a guidance of this research process.

Conduct Background Reading

Utilize primary reference and materials such as books in mathematical economics and operations research.

Search for Books

- Primary Books: Standard mathematical economics books published by McGraw-Hill Inc and operations research books published by Holden-Day Inc. Use the library catalog to find both print and electronic books.

Evaluate Sources

Critically assess the credibility and relevance of each source:

- Authority: Authors of the books and their expertise
- Publication Date
- Relevance: The source directly addressed research question
- Citations: The source and citation should be reputable works. This evaluation ensures the reliability and quality of the information incorporate into this research

Organize Findings

Systematically arrange notes and sources manually in the frame of inductive-and-deductive method. Findings found from problem-solving investigation were organized in the frame of the bipolar method. Before being quantitatively synthesized and analyzed the investigation results were grouped in some tables.

Synthesize, Analyze, and Infer Conclusion
Integrate information from problem-solving process to develop a coherent understanding of this research topic. Analyze quantitatively problem-solving problems and findings to construct a well-rounded argument, and finally conclude findings in the frame of inductive-and-deductive method.
Check Citation
The purpose of this step is for qualitatively determine whether the sources of research are relevant and appropriate for investigation purpose
Present Research Result
This final step is to compile findings and conclusion into a structured format. Clearly communicate methodology, analysis, and conclusions ensuring that this scholarly work is accessible and comprehensible to intended readers.

RESULT AND DISCUSSION

To get a result, it is necessary to do investigation procedure that is a step-by-step guide for conducting explorations, making conjectures, and verification for getting research question answers in the frame of inductive-deductive method. The investigation procedure described as follows:

- Choose appropriate economic functions for solving-problem processes and explorations.
- Some derivative formulas can satisfy the necessary and sufficient conditions for each solution acquired in studying constrained optimization problems.
- Test the derivative formulas whether (or not) they are the appropriate tools for helping solving-problems in mathematical economics.
- The advantage and limitations of mathematics in helping solve constrained and non-constrained optimization problems in economics or business.

To obtain optimum results in mathematical calculations, it is necessary to fulfill the following conditions:

- Necessary conditions must meet the first derivative order.
- Sufficient conditions must fulfill the second derivative order

Derivative formula:

$f(x) = ax^n + bx + c$ example: $f(x) = 2x^3 + 3x + 6$

$f'(x) = anx^{n-1} + b$ $f(x) = 6x^2 + 3$

$f''(x) = 12x$

$y = ax_1^n x_2 + bx_1x_2 + c$ example: $y = f(x_1,x_2) = 2x_1^3x_2 + 3x_1x_2 + 5$

$y_1 = dy/dx_1 = anx_1^{n-1}x_2 + bx_2$ $f_1 = df/dx_1 = 6x_1^2x_2 + 3x_2$

$y_2 = dy/dx_2 = ax_1^n + bx_1$ $f_2 = df/dx_2 = 2x_1^3 + 3x_1$

$f_{11} = d^2f/dx_1^2 = 12x_1x_2$

$f_{22} = d^2f/dx_2^2 = 0$

$f_{12} = f_{21} = d^2f/dx_1dx_2 = d^2f/dx_2dx_1 = 6x_1^2 + 3$

Table 1. Conditions for the extreme: $Z = f(x_1,x_2,\dots,x_n)$

Conditions	Maximum	Minimum
First order	$f_1 = f_2 = f_3 = \dots = f_n = 0$	$f_1 = f_2 = f_3 = \dots = f_n = 0$
Second order	$\lvert H_1\rvert < 0; \lvert H_2\rvert > 0;$ $\lvert H_3\rvert < 0;\dots.$ or d^2z definite negative	$\lvert H_1\rvert; \lvert H_2\rvert; \lvert H_n\rvert > 0$ or d^2z definite positive

Adapted from: Chiang, 1982

Conditions for constrained extreme: $Z = f(x_1,x_2,\dots,x_n)$ that satisfied constraint function $g(x_1,x_2,\dots,x_n) = c$ with $Z = f(x_1,x_2,\dots,x_n) + \dots(c - g(x_1,x_2,\dots,x_n))$

Table 2. Conditions for the constrained extreme: $Z = f(x_1,x_2,\ldots,x_n)$

Conditions	Maximum	Minimum
First order	$Z_{\ldots} = Z_1 = Z_2 = \ldots = Z_n = 0$ or $dz = 0$ attributed to $g = c$	$Z_{\ldots} = Z_1 = Z_2 = \ldots = Z_n = 0$ or $dz = 0$ attributed to $g = c$
Second order	$\lvert\bar{H}_2\rvert > 0; \lvert\bar{H}_3\rvert < 0; \lvert\bar{H}_4\rvert > 0$ or d^2z is a definite negative, attributed to $dg = 0$	$\lvert\bar{H}_2\rvert, \lvert\bar{H}_3\rvert, \ldots, \lvert\bar{H}_n\rvert < 0$ or d^2z is a definite positive, attributed to $dg = 0$

Adapted from: Chiang, 1982

Karush-Kuhn-Tucker requirement (KKT)

KKT will be appropriate for solving a general optimization problem, especially in a constrained optimization problem with non-linear functions.

Theorem: If $f(x)$, $g_1(x)$, $g_2(x)$, … ,$g_m(x)$ as a function that has derivative and satisfy regulation, then $x^* = (x_1^*, x_2^*, \ldots, x_n)$ as an optimal solution for non-linear programming if there is *m, u, u, … , u* so all of the following requirements can be satisfied:

1. $\frac{df}{dx_j} - \sum_{i-1}^{m} u_i \frac{dg_i}{dx_j} \leq 0$
2. $x_j^* \left(\frac{df}{dx_j} - \sum_{i-1}^{m} u_i \frac{dg_i}{dx_j} \right) = 0$

   Requirement (1) and (2) above at x=x for j=1,2,3,...,n*
3. $g_i(x^*) - b_i \leq 0$
4. $u_i\,(g_i(x^*) - b_i) = 0$
   Requirement (3) and (4) for i=1,2,3,…,m

5. $x_j^* \geq 0$
   Requirement (5) for j=1,2,3,…,n
6. $u_i \geq 0$
   Requirement (6) for i=1,2,3,…,m

KKT requirements, as mentioned above, are not enough to obtain an optimal solution. Requirements (3) and (5) do not ensure a feasible solution area, whereas other requirements can make a possible solution to be an optimal solution. But if it can satisfy requirement (2), it cannot ensure an optimal solution.

In the following table, we added an assumption of convex conditions. We need this assumption for ensuring to get an optimal solution. As the effect of adding the assumption, we now obtain the following theorem as the broadening of the above theorem, namely:

> If $f(x)$ is a concave function and $g_1(x)$, $g_2(x)$, … ,$g_m(x)$ are convex functions *(case in convex programming*) and all functions satisfy regulation, then $x^* = (x^*_1, x^*_2, \ldots, x^*_n)$ as *the optimal solution* if and only if *all conditions* in the former theorem are satisfied (Hillier & Lieberman, 1980).

Table 3. Necessary and sufficient conditions in optimization

Problems	Necessary conditions	Sufficient conditions
One variable function without constraints	$df/dx = 0$	*f(x) is concave*
Multi-variable function without constraints	$df/dx_i = 0$ $(j = 1, 2, \ldots, n)$	*f(x) is concave*
Non-negative constraints optimization	$df/dx_j \leq 0$ and $x_j(df/dx_j) = 0$ $(j = 1, 2, \ldots, n)$	

General constrained optimization	*Requirements of Karush-Kuhn-Tucker*	*f(x) is concave* *$g_i(x)$ is convex (i = 1, 2, …,m)*

Adopted from: Hillier & Lieberman (1980)

- Convex if and only if $d^2f(x)/dx^2 \geq 0$ for all possible value of x
- Strictly convex if and only if $d^2f(x)/dx^2 > 0$ for all possible value of x
- Concave if and only if $d^2f(x)/dx^2 \leq 0$ for all possible value of x
- Strictly concave if and only if $d^2f(x)/dx^2 < 0$ for all possible value of x

The following figures show some possibilities of the graphs of convex and concave functions. (Adapted from Hillier & Lieberman, 1980)

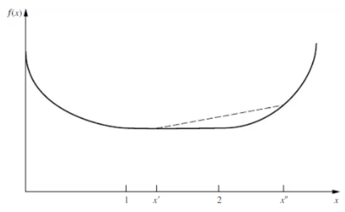

Fig 1. A convex function

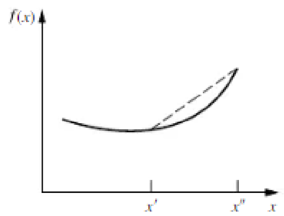

Fig 2. A strictly convex function

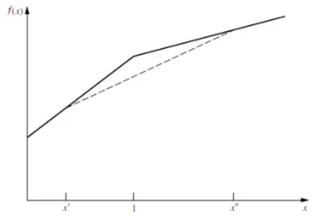

Fig 3. A Concave Function

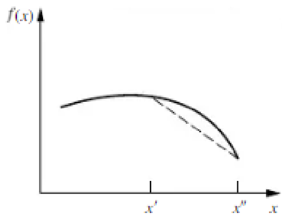

Fig 4. A strictly concave function

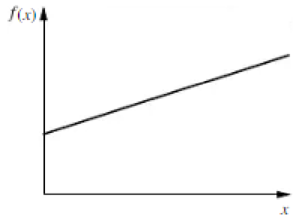

Fig 5. Both convex and concave functions

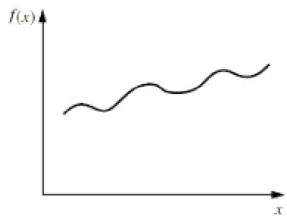

Fig 6. Neither convex nor conconcave functions

The investigation method is a necessary part of this investigation. This investigation used the deductive method for mathematical proof and the inductive method for mathematical explorations, such as in experimental mathematics. The inductive and deductive methods will be described in one section, namely: Inductive-Deductive method

The explorations consist of solving problems on constrained optimization problems in the form of mathematical economics functions. These explorations and investigations will be described in three sections as follows:

- Substitution Method
- Lagrange Multiplier Method
- KKT requirements in the application

Due to the importance of the two methods in this investigation, we need to combine inductive and deductive methods. The combination of these two methods in this investigation can be described as a researcher moving between the inductive and deductive poles – bipolar methods.

> *"This has become the main characteristic of new research method. In this new method researcher moves to and from between the poles of inductive and deductive such as precisely mentioned by John Dewey 'Reflective Thinking'." (Sutrisno, 1980)*

This reseach study used and approached the combination of inductive and deductive methods such as described below:

Inductive approach emphasizes the activity of the investigation first, which is continued by drawing conclusions based on the analysis. As stated above, this approach is often referred to as concluding the particular to the general. Inductive research will be carried out when the reference source is not found on a topic, and inductive study is often carried out because there is no theory that can be tested.

In deductive method, if truth has been generally understood, it will be able to achieve the status of new knowledge about a particular issue or indication. When a deduction is concluded, the deduction is a thinking activity based on general things (theories, concepts, principles, beliefs) that leads to special, based on general things that lead to specific conclusions, which are the elements of a problem or incident.

When conducting deductive research, researchers should always start with theory (inductive research results). The purpose of deductive reasoning is to test theories, general ideas, or hypotheses; if there is no one, then researchers cannot conduct deductive research.

Inductive Step

Inductive approach consists of three steps, and the following are the steps and examples of inductive research, namely: case-by-case observation, pattern observation, generalization, or theory development.

The limitation of inductive method:

- *Conclusion or generalization acquired is difficult to or even never to be proved.*
- *The conclusion made is easily cancelled.*

Deductive Step

The deductive research approach consists of four steps. Below are the steps with an example of deductive research: start from the existing theory, collect data for analysis, and infer conclusions.

The limitation of deductive method

- *Conclusions from deductive reasoning can only be valid if all the premises set out in an inductive study are valid.*

Purpose of Inductive Research

The understanding of the inductive research method is a method of process thinking starting from something specific leading to the general, where in making conclusions using observations. Inductive research aims to find new knowledge. It can be formed on something that is of interest to researcher. The researcher will determine the research problem based on what is being done and formulate research questions.

Furthermore, the researcher will try to obtain the data. Researcher can use various research methods to collect data as the material for the base of a research question. The collection can be done through observation, interviews, etc.

In the analysis step, the researcher will observe a particular pattern from the data that has been collected. Meanwhile, in the final part of the inductive research, the researcher will develop a theory using the patterns and data found. Based on the Grounded theory (expressed by Glaser and Strauss), the essence of discovering new knowledge will proceed in a cyclic manner using a bottom-up approach.

Purpose of Deductive Research

The definition of the deductive research method is a method of thinking activity starting from something general that leads to the specific and using logic to conclude. Deductive research has its way of carrying out the process, and it uses a top-down approach. If further understood, deductive research has categories for hypothesis testing activity that aims to validate a theory. It is different from inductive research; in creating new knowledge, inductive research is more about testing a theory.

Deductive research does not seek to find patterns in the data but uses observation to verify a pattern. Researchers use this method to manipulate theory. The deductive approach is very familiar with quantitative research, in which the researcher will attempt to find causation and present analysis.

For implementing the bi-polar method in this investigation, this investigation explored the use of the substitution method, Lagrange multiplier method, and KKT requirements in a constrained optimization problem in economics such as described below.

Substitution Method

As mentioned above, solving the optimization problem, a straightforward one, it can be done using the substitution method. Our discussion begins by considering the utility function $U = 3Q_1Q_2 + 3Q_2$ faced by a group of customers (by adding a budget constituency owned by this group). If the group provides a budget of *$ US 100* for buying both types of goods (Q_1 and Q_2) while the prices of Q_1 and Q_2 are *$ 20* and *$ 10*, respectively, then the linear equation for the budget is $20Q_1 + 10Q_2 = 100$. (Yaqin, 2008)

To find the optimal value of the simple problem above, we can still use the following techniques:

$$U = 3Q_1Q_2 + 3Q_2 \quad \ldots\ldots(1)$$

$$20Q_1 + 10Q_2 = 100 \quad \leftrightarrow \quad 2Q_1 + Q_2 = 10$$

$$Q_2 = 10 - 2Q_1 \quad \ldots\ldots(2)$$

Now we substitute equation (2) into equation (1) to get

$U = 3Q_1\ (10 - 2Q_1) + 3(10 - 2Q_1)$
$= 30Q_1 - 6Q_1^2 + 30 - 6Q_1$
$= - 6Q_1 + 24Q_1 + 30$

From $U = - 6Q_1^2 + 24Q_1 + 30$ we can get first-order derivative, namely:

$U_1 \equiv dU/dQ_1 = - 12Q_1 + 24$

For satisfying U-maximum, its necessary condition is $dU/dQ_1 = 0$ namely:

$- 12Q_1 + 24 = 0$

$Q_1 = 2$ ……………………………………………………(3)

For looking for the value of Q_2, we need to substitute *(3) i*nto equation *(2)* as follows:

$Q_2 = 10 - 2(2)$
$= 6$ ………………………………………………………….(4)

Then, substitute the values of *(3) and (4)* into equation *(1),* namely:

$U = 3(2)(6) + 3(6)$
$= 54$

Thus, with the values $Q_1 = 2$ and $Q_2 = 6$, we can get the value of constrained optimum $U = 54$

Lagrange Multiplier Method

Another method that can be used to solve the optimization problem with a more complex constraint function is the Lagrange (indeterminate) multiplier method. The essence of this method is to change the form of a function in such a way that the first-order conditions of the free extremum problem can still be used to solve a constrained optimization problem.

To find out that the Lagrange Multiplier method is more general than the technique or solution method above, it is appropriate if we use this method to discuss or solve the same problem (as mentioned above).

As has been shown (discussed) above, maximizing the value of *U* on the utility function $U = 3Q_1Q_2 + 3Q_2$ bound to the $20Q_1 + 10Q_2 = 100$ constraint is an optimization problem successfully solved using relatively simple techniques like the substitution method. However, to discover and prove the general nature of the Lagrange Multiplier method, let us try to solve the problem above using the Lagrange multiplier method.

The first step that needs to be done is to form a Lagrange function which is a modified version of the objective function *U* (as shown above) which is combined with its constraints in the following form:

λ is a symbol that is an undetermined number, which is called a Lagrange (indeterminate) multiplier. In this case, if the constraint can be satisfied, regardless of the value of λ, then the last term in the above equation will be lost so that the *U* function will be the same as the *Z* function. Thus, we can perform optimization without having to be disturbed again by the constraints.

However, the problem is how to engineer the Lagrange function so that its constraint loses from the *Z* function (because the constraint has been met). In other words, we can run a free optimization process on the *U* function as compensation for the constrained optimization in connection with fulfilling the constraint.

The technique to obtain the expected result is enough to assume, that is an additional variable in the *Z* function, namely $Z = z\ (\lambda, Q_1, Q_2)$. Thus, the first-order conditions (necessary conditions) for the free extremes will consist of the following sets of simultaneous equations.

$$Z_\lambda \equiv {}^{dz}\!/_{d\lambda} = 100 - 20Q_1 - 10Q_2 = 0 \ldots\ldots\ldots\ldots(1)$$
$$Z_1 \equiv {}^{dz}\!/_{dQ_1} = 3Q_2 - 20\lambda = 0 \ldots\ldots\ldots\ldots(2)$$
$$Z_2 \equiv {}^{dz}\!/_{dQ_2} = 3Q_1 + 3 - 10\lambda = 0 \ldots\ldots\ldots\ldots(3)$$

If equations (1), (2), and (3) be solved simultaneously, we can acquire the values of:
$Q_1 = 2$, $Q_2 = 6$, and $\lambda = 9/10$

The following calculation can get the above values:

$100 - 20Q_1 - 10Q_2 = 0$ ....(1)

$3Q_2 - 10\lambda = 0 \leftrightarrow 3Q_2 = 20\lambda \leftrightarrow Q_2 = 20\lambda/3$ ....(4)

$3 + 3Q_1 - 10\lambda = 0 \leftrightarrow 3Q_1 = -3 + 10\lambda$

$\leftrightarrow Q_1 = (-3 + 10\lambda)/3$ ....(5)

Now we substitute (4) and (5) into (1)

$100 - 20(-3 + 10\lambda)/3 - 10(20\lambda/3) = 0$

$\leftrightarrow 100 + 60/3 - 200\lambda/3 - 200\lambda/3 = 0$

$\leftrightarrow 300/3 + 60/3 - 200\lambda/3 - 200\lambda/3 = 0$

$\leftrightarrow 360/3 = 400\lambda/3$

$\leftrightarrow 1200\lambda = 1080$

$\leftrightarrow \lambda = 9/10$

Substitute (6) into (4) : $Q_2 = [20(9/10)]/3 = 6$

Substitute (6) into (5): $Q_1 = [-3 + 10(9/10)]/3 = 2$

By substituting the values of $Q_1 = 2$, $Q_2 = 6$ (and $\lambda = 9/10$) into equation *Z*, we will get the value of function Z as the optimum value of function Z, namely $U = 54$.

The calculations for getting the value of $U = 54$ are as follows: $Z = 3(2)(6) + 3(6) = 54$

However, sufficient conditions must be met first to determine whether the obtained values of Q_1 and Q_2 are indeed the values (outputs) that give a maximum value of *54*. As has been done previously, the second order derivative, in this case, can also be expressed as a determinant. Its value must be determined as a step to determine the sufficient conditions for the optimization process results $U = 54$ above.
For function prevails, the requirements are as follows:

$$Z = f(x,y) + \lambda[c - g(x,y)]$$

d^2z is definite positive if $|\overline{H}| < 0$
d^2z is definite negative if $|\overline{H}| > 0$

In where:

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix}$$

$|\overline{H}| =$ Determinant Hesse

f(x,y) and $g(x,y) = c$ are the objective function and constraint function respectively.
Finally, let us now investigate whether the value of $Z = 54$ $(U = 54)$ was obtained because the outputs $Q_1 = 2$ and $Q_2 = 6$ are optimal results. The second order derivative of the Z function is:

$Z_{11} = 0$ $Z_{12} = 3$ and from $20Q_1 + 10Q_2 = 100$
$Z_{21} = 3$ $Z_{22} = 0$ obtained $g_1 = 20$ and $g_2 = 10$

$$|\overline{H}| = \begin{vmatrix} 0 & 20 & 10 \\ 20 & 0 & 3 \\ 10 & 3 & 0 \end{vmatrix} = (0 + 600 + 600) - (0 + 0 + 0) = 1200 > 0$$

Thus, we find $|\overline{H}| > 0$

KKT requirements in the application

The following paragraph is the problem-solving of constrained optimization using the formulation and application of KKT requirement. This problem is *adapted from Hillier & Lieberman (1980),* that described the view of exploration of the constrained optimization problem in non-linear programming with two variables.

Maximize : $f(x) = ln(x_1+1) + x2$
Constraints : $2x_1 + x_2 < 3$, $x_1 > 0$, $x_2 > 0$

$f(x)$ is natural logarithm as a concave function and for $m = 1$ and $g(x) = 2x_1 + x_2$ that cause $g(x)$ is convex. Thus, KKT requirements, as mentioned above, can be used to look for an optimal solution.
Its conditions are as follows:

1(a). $1/(x_1+1) – 2u_1 \leq 0$
1(b). $1 – u_1 < 0$
2(a). $x_1[1/(x_1+1) – 2u_1] = 0$
2(b). $x_2(1 – u_1) = 0$
3. $2x_1 + x_2 < 3$
4. $u_1(2x_1 + x_2 – 3) = 0$
5. $x_1 > 0$, $x_2 > 0$
6. $u_1 > 0$

Furthermore, we simplify the KKT requirements become as follows:

1. $u_1 \geq 1$ *…..… from 1(b)*
   $x_1 \geq 0$ *…..… from 5*
2. *Furthermore:* $1/(x_1+1) – 2u_1 < 0$
3. *Then,* $x_1 = 0$ *…..… from 2(a)*
4. *If* $u_1 \neq 0$ *then* $2x_1 + x_2 – 3 = 0$ *….… from (4)*
5. *Step 3 and 4 affect* $x_2 = 3$
6. *If* $x_2 \neq 0$ *then* $u_1 = 1$ *…..… from 2(b)*
7. *Hence, from the information above we get:* $x_1 = 0$, $x_2 = 3$, $u_1 = 1$

Acquired are $u_1 = 0$ (satisfy all requirements), so $x_1 = 0$ dan $x_2 = 3$
Thus $x^* = (0,3)$ as an optimal solution for the example above.

Mathematical formulas can help economists solve economic problems and formulate the solving problem for obtaining optimal solutions. But we must understand that mathematical programs generally work on continuous variables, so perhaps we got an unexpected number or solution like 15,73 cars in practice. In other words, we cannot get an integer of the number of cars as an optimal solution. Fortunately, this problem can be solved mathematically by integer programming that only results in a solution with integer numbers (Chiang, 1982). So far, we have studied constrained optimization with its problem and solution being static. Otherwise, we must build the *optimal time path* for each variable chosen in a dynamic optimization problem. Still, this article will not study and solve the type of dynamic optimization problems.

Investigation Result

Explorations with calculating and solving on the example of the utility function, we found results as follows:

- $|\overline{H}| > 0$ then *54* is the optimal (maximum) result in the constrained optimization above.
- Results of calculations as shown in Table 4 below.

Table 4. Results of Explorations for $U = 3Q_1Q_2+3Q_2$

N	Q_1	Q_2	$2Q_1 + Q_2 = 10$	*U*	*Explanation*
1	**0**	**10**	**10**	**30**	
2	**1**	**8**	**10**	**48**	
3	**2**	**6**	**10**	**(54)**	**Optimal**
4	3	4	10	48	
5	4	2	10	30	
6	5	0	10	0	

$U = 3Q_1Q_2+3Q_2$

Thus: Due to $|\overline{\mathrm{H}}| > 0$, then *54* is the optimal (maximum) result in the constrained optimization.

Exploration Results with KKT requirements:

The following paragraph presents the results of KKT exploration for solving the optimization problem; and the process as summarized below:

Maximize : $f(x) = ln(x_1+1) + x2$
Constraints : $2x_1 + x_2 < 3$, $x_1 > 0$, $x_2 > 0$
From the exploration we get: $x_1 = 0$, $x_2 = 3$, $u_1 = 1$
Furthermore, we acquired are $u_1 = 0$ (satisfy all requirements), so $x_1 = 0$ dan $x_2 = 3$
Thus $x^* = (0,3)$ as an optimal solution.

CONCLUSION

We already know that mathematics (in this case, derivatives or differential formulas) has helped us a lot in solving various problems in business and economics, mainly to solve the optimization problem. It must be understood that the real issues related to the optimization problem must be first translated into the mathematical language; in other words, a mathematical model must be created first so that mathematical formulas can be used to solve the problem.

Especially for solving or solving problems with optimization constraints, we can approach and modify the mathematical functions linearly or non-linearly so that derivative formulas can be used. Modification of these functions can be done in several ways, including:

1. Substitution Method
2. Lagrange Multiplier Method
3. KKT requirements

It could be very satisfying if our solving problem could satisfy both necessary and sufficient conditions (Chiang, 1984). But the weakness is likely we cannot succeed in catching an optimal solution due to we failed to meet the sufficient conditions.

Mathematical formulas can help economists solve economic problems and formulate the solving problem for obtaining optimal solutions. But we must understand that mathematical programs generally work on continuous variables, so perhaps we may get an unexpected number for a solution is possible, like 15,73 units of cars. In other words, we cannot always hope to get the integer of the number of cars as an optimal solution. Fortunately, this limitation can be solved mathematically by integer programming that can only generate a solution with integer numbers (Chiang, 1982).

The problem formulated and discussed above is just a general case of constrained optimization with non-negative constraints. As for general problems with constraints – which have been translated into the objective function f(x) with g(x) as its constraint function, Karush-Kuhn-Tucker method (KKT) could be helpful (Hillier & Lieberman, 1980). The necessary and sufficient conditions that must be met are:

1) Necessary conditions : Karush-Kuhn-Tucker (KKT) requirements for constrained optimization.
2) Sufficient conditions : $f(x)$ is concave and $g_i(x)$ is *convex* $(i = 1, 2, ..., m)$

So far, we have worked on static problems and solutions that do not relate to time. Otherwise, we must build an optimal time path for dynamic problems and solutions. We need further research to solve the optimization type and a higher level of derivative calculus like variation calculus and dynamic programming.

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